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THE 15 SUBCASES OF THE INTERRELATIONSHIPS OF THE COEFFICIENTS IN THE KERNEL OF A CONTINUOUS STRICTLY CONVEX ANTAGONISTIC GAME AND THE NINE FIGURED OUT CORRESPONDING TYPES OF THE SOLUTION

As for the marketing dual activity problem there have been investigated the 15 subcases of the interrelationships of the coefficients in the kernel of a continuous strictly convex antagonistic game. From that the corresponding nine game solution types have been determined.

Як для задачі маркетингової дуальної діяльності досліджено 15 підвипадків співвідношень коефіцієнтів у ядрі однієї неперервної строго опуклої антагоністичної гри. З цього визначено відповідні дев'ять типів розв'язків гри.

Key words: marketing dual activity, strictly convex antagonistic game, game solution type, solutions continuum.

The issue specification in the presented paper and the investigation goal designation with the game kernel

An antagonistic game gives a well fitting mathematical model for making decisions in some competitive activity economical processes (like the marketing activity problem), where the two players are the rivals. An actual investigation goal designation in the antagonistic game stays in finding all the solutions

$$\mathscr{S} = \left\{ \mathscr{X}_{\text{opt}}, \mathscr{Y}_{\text{opt}}, V_{\text{opt}} \right\} \tag{1}$$

of the strictly convex continuous antagonistic game [1 — 3], which kernel

$$S(x, y) = ax^2 + bx + gxy + cy + hy^2 + k \tag{2}$$

as the surface is defined generally on the unit square

$$D_S = X \times Y = [0; 1] \times [0; 1], \tag{3}$$

where an element $x \in X = [0; 1]$ is a pure strategy of the first player, and an element $y \in Y = [0; 1]$ is a pure strategy of the second player. There in the formula (1) the denomination V_{opt} is assigned for the game value. The optimal strategies set of the first player has been denoted as $\mathscr{X}_{\text{opt}}$, and the optimal strategies set of the second player has been denoted as $\mathscr{Y}_{\text{opt}}$. This paper investigation goal designation is to find all the solutions (1) of the continuous strictly convex antagonistic game with the kernel (2), which is defined on the unit square (3), where the kernel coefficients $a > 0$, $b < 0$, $g > 0$, $c < 0$ and the constant $k \in \mathbb{R}$. As this game has been said to be the strictly convex, then $\forall\ x \in X$ and $\forall\ y \in Y$ there must be held the condition

$$\frac{\partial^2 S(x, y)}{\partial y^2} > 0, \tag{4}$$

whence

$$\frac{\partial^2 S(x, y)}{\partial y^2} = 2h > 0 \tag{5}$$

and the coefficient by the square of a pure strategy of the second player $h > 0$.

The total thorough solving of the specified continuous strictly convex antagonistic game with the adjusted kernel

When solving the specified continuous strictly convex antagonistic game with the adjusted kernel (2), there should be applied the known maximin method [4] with the total determining the sets $\mathscr{X}_{\text{opt}}$ and $\mathscr{Y}_{\text{opt}}$ [5, 6]. First of all, there should be marked, that as the coefficient $a > 0$ then the parabola (2), being the function of the only variable x, does not have the global maximum point. Consequently this parabola on the unit segment $X = [0; 1]$ reaches the maximum either in the point $x = 0$ or $x = 1$ and, certainly, this maximum depends on the sign of the statement $a + b + gy$. While having $a > 0$, $b < 0$, $g > 0$, $c < 0$ then there $a + b + gy > 0$ if $y > -\frac{a+b}{g}$. The value $-\frac{a+b}{g} > 0$ when $a + b < 0$, and the value $-\frac{a+b}{g} < 1$ when $a + b + g > 0$. That will be the opening two subcases in the specified game investigation.

Subcase 1.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g>0$. As here $a+b+gy>0 \quad \forall\ y>-\dfrac{a+b}{g}$ by $-\dfrac{a+b}{g}\in(0;1)\subset Y$ then the maximum of the surface (2) on the unit segment X of the variable x is

$$\max_{x\in X} S(x,\ y)=\begin{cases} S(0,\ y)=cy+hy^2+k,\ y\in\left[0;\ -\dfrac{a+b}{g}\right], \\ S(1,\ y)=a+b+gy+cy+hy^2+k,\ y\in\left[-\dfrac{a+b}{g};\ 1\right]. \end{cases} \tag{6}$$

For minimizing the function (6) on the unit segment Y it is necessary that the minimum of the parabola $S(0,\ y)$ on the segment $\left[0;\ -\dfrac{a+b}{g}\right]\subset Y$ should be determined, and also that the minimum of the parabola $S(1,\ y)$ on the segment $\left[-\dfrac{a+b}{g};\ 1\right]\subset Y$ should be determined.

Before finding the local minimum of the parabola $S(0,\ y)$ on some subsegment of the unit segment Y primarily the global minimum of the parabola $S(0,\ y)$ should be determined. The first derivative of the parabola $S(0,\ y)$ is

$$\frac{d}{dy}S(0,\ y)=\frac{d}{dy}\left(cy+hy^2+k\right)=c+2hy \tag{7}$$

and the first critical point of the parabola $S(0,\ y)$ is the zero point of the line (7) $y=y_{\text{cr}}^{\langle 0\rangle}=-\dfrac{c}{2h}$. The second derivative of the parabola $S(0,\ y)$ is

$$\frac{d^2}{dy^2}S(0,\ y)=\frac{d}{dy}(c+2hy)=2h>0 \tag{8}$$

and the global minimum of the parabola $S(0,\ y)$ is

$$y_{\text{cr}}^{\langle 0\rangle}=y_{\min}^{\langle 0\rangle}=-\frac{c}{2h}. \tag{9}$$

However, by the initial conditions $c<0$ and $h>0$, here the point $y_{\min}^{\langle 0\rangle}=-\dfrac{c}{2h}>0$.

The first derivative of the parabola $S(1,\ y)$ is

$$\frac{d}{dy}S(1,\ y)=\frac{d}{dy}\left(a+b+gy+cy+hy^2+k\right)=g+c+2hy. \tag{10}$$

The first critical point of the parabola $S(1,\ y)$ is the zero point of the line (10), that is $y=y_{\text{cr}}^{\langle 1\rangle}=-\dfrac{g+c}{2h}$, and as the second derivative of the parabola $S(1,\ y)$ is

$$\frac{d^2}{dy^2}S(1,\ y)=\frac{d}{dy}(g+c+2hy)=2h>0 \tag{11}$$

once again, then the global minimum of the parabola $S(1,\ y)$ is

$$y_{\text{cr}}^{\langle 1\rangle}=y_{\min}^{\langle 1\rangle}=-\frac{g+c}{2h}. \tag{12}$$

For ascertaining whether $y_{\min}^{\langle 0\rangle}\in\left[0;\ -\dfrac{a+b}{g}\right]$ or not, will determine the following difference:

$$y_{\min}^{\langle 0\rangle}-\left(-\frac{a+b}{g}\right)=-\frac{c}{2h}-\left(-\frac{a+b}{g}\right)=-\frac{c}{2h}+\frac{a+b}{g}=\frac{2h(a+b)-cg}{2hg}. \tag{13}$$

Herewith $y_{\min}^{\langle 0\rangle} < -\dfrac{a+b}{g}$ by $2h(a+b)-cg<0$ and $y_{\min}^{\langle 0\rangle} \geqslant -\dfrac{a+b}{g}$ by $2h(a+b)-cg \geqslant 0$.

Subcase 1.1.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g>0$; $2h(a+b)-cg<0$. Having the point $y_{\min}^{\langle 0\rangle} \in \left(0; -\dfrac{a+b}{g}\right)$ and the difference

$$y_{\min}^{\langle 0\rangle} - y_{\min}^{\langle 1\rangle} = -\frac{c}{2h} - \left(-\frac{g+c}{2h}\right) = -\frac{c}{2h} + \frac{g+c}{2h} = \frac{g}{2h} > 0 \qquad (14)$$

displaying that $y_{\min}^{\langle 0\rangle} > y_{\min}^{\langle 1\rangle}$, else get that $y_{\min}^{\langle 1\rangle} < -\dfrac{a+b}{g}$. Then there is the double parabolic inequality with the point (12)

$$S\left(1,\, y_{\min}^{\langle 1\rangle}\right) = S\left(1, -\frac{g+c}{2h}\right) < S\left(1, -\frac{a+b}{g}\right) < S(1, 1). \qquad (15)$$

Also there is the obvious identity

$$S\left(1, -\frac{a+b}{g}\right) = S\left(0, -\frac{a+b}{g}\right). \qquad (16)$$

The double parabolic inequality (15) and the identity (16) with $y_{\min}^{\langle 0\rangle} \in \left(0; -\dfrac{a+b}{g}\right)$ rule for that the minimum of the function (6)

$$\min_{y\in Y}\max_{x\in X} S(x, y) = \min\left\{ \min_{y\in\left[0;\, -\frac{a+b}{g}\right]} S(0, y),\ \min_{y\in\left[-\frac{a+b}{g};\, 1\right]} S(1, y) \right\} =$$

$$= \min\left\{ S\left(0,\, y_{\min}^{\langle 0\rangle}\right),\ \min\left\{ S\left(1, -\frac{a+b}{g}\right),\ S(1, 1)\right\}\right\} = \min\left\{ S\left(0, -\frac{c}{2h}\right),\ S\left(1, -\frac{a+b}{g}\right)\right\} =$$

$$= \min\left\{ S\left(0, -\frac{c}{2h}\right),\ S\left(0, -\frac{a+b}{g}\right)\right\} = S\left(0, -\frac{c}{2h}\right) = S\left(0,\, y_{\min}^{\langle 0\rangle}\right) = c\left(-\frac{c}{2h}\right) + h\left(-\frac{c}{2h}\right)^2 + k = k - \frac{c^2}{4h} = V_{\text{opt}} \qquad (17)$$

is reached in the point $y = y_{\text{opt}} = -\dfrac{c}{2h}$, that is on the set of the second player optimal strategies

$$\mathscr{Y}_{\text{opt}} = Y_{\text{opt}} = \left\{-\frac{c}{2h}\right\} = \left\{y_{\text{opt}}\right\}, \qquad (18)$$

which coincides with the second player optimal pure strategies set Y_{opt}. The set of the first player optimal pure strategies X_{opt} primarily should be determined by the roots x_1 and x_2 of the quadratic equation [1 — 6]

$$V_{\text{opt}} = S\left(x,\, y_{\text{opt}}\right). \qquad (19)$$

Hereon the corresponding equation (19) is

$$V_{\text{opt}} = S\left(0, -\frac{c}{2h}\right) = k - \frac{c^2}{4h} = ax^2 + bx + gx\left(-\frac{c}{2h}\right) + c\left(-\frac{c}{2h}\right) + h\left(-\frac{c}{2h}\right)^2 + k = ax^2 + bx +$$

$$+gx\left(-\frac{c}{2h}\right) + k - \frac{c^2}{4h} = x\left(ax + b - \frac{cg}{2h}\right) + k - \frac{c^2}{4h} = x\left(ax + \frac{2hb-cg}{2h}\right) + k - \frac{c^2}{4h} = S\left(x, -\frac{c}{2h}\right) = S\left(x,\, y_{\text{opt}}\right). \qquad (20)$$

From the equation (20) get the equation

$$x\left(ax + \frac{2hb-cg}{2h}\right) = x\left(x + \frac{2hb-cg}{2ah}\right) = 0, \qquad (21)$$

where the roots of the equation (19) are $x_1 = 0$ and $x_2 = \dfrac{cg-2hb}{2ah}$. But the initial condition $2h(a+b)-cg<0$

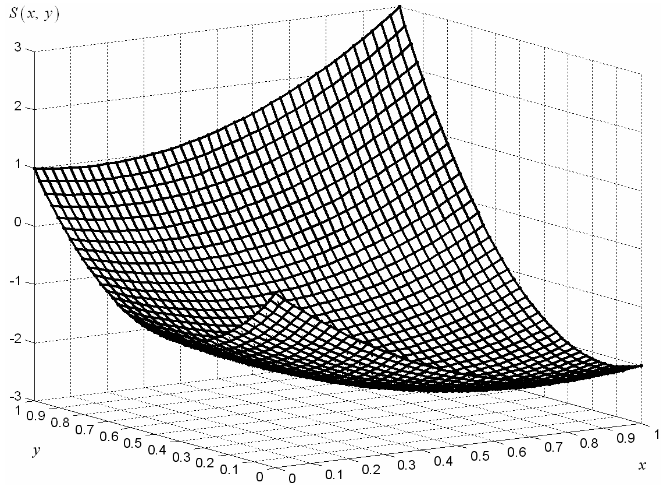

Fig. 1. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g>0$, $2h(a+b)-cg<0$ with the kernel $S(x,\, y)=3x^2-5x+4xy-6y+7y^2$ and solution $\mathscr{S}=\left\{\{0\},\, \left\{\frac{3}{7}\right\},\, -\frac{9}{7}\right\}$

means that $x_2=\dfrac{cg-2hb}{2ah}>1$. So there be $x_1\in X$ and $x_2\notin X$. Thereupon here is the set

$$X_{\text{opt}}=\{x_1\}=\{0\}=\mathscr{X}_{\text{opt}} \qquad (22)$$

and the investigated subcase (fig. 1) game solution is the set

$$\mathscr{S}=\left\{\{0\},\, \left\{-\frac{c}{2h}\right\},\, k-\frac{c^2}{4h}\right\}. \qquad (23)$$

Subcase 1.1.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g>0$; $2h(a+b)-cg\geqslant 0$. As the point $y_{\min}^{\langle 0\rangle}\geqslant -\dfrac{a+b}{g}$ then there is the double parabolic inequality with the point (9)

$$S(0,\, 0)>S\left(0,\, -\frac{a+b}{g}\right)\geqslant S\left(0,\, -\frac{c}{2h}\right)=S\left(0,\, y_{\min}^{\langle 0\rangle}\right). \qquad (24)$$

Further will clear out if $y_{\min}^{\langle 1\rangle}\in\left(-\dfrac{a+b}{g};\, 1\right]$. Here the point $y_{\min}^{\langle 1\rangle}\leqslant 1$ by $g+c+2h\geqslant 0$, but for ascertaining whether $y_{\min}^{\langle 1\rangle}\in\left(-\dfrac{a+b}{g};\, 1\right]$ or not, will determine the following difference:

$$-\frac{g+c}{2h}-\left(-\frac{a+b}{g}\right)=-\frac{g+c}{2h}+\frac{a+b}{g}=\frac{2h(a+b)-g(g+c)}{2hg}. \qquad (25)$$

Now see that $y_{\min}^{\langle 1\rangle}\in\left(-\dfrac{a+b}{g};\, 1\right]$ by the conditions $2h(a+b)-g(g+c)>0$ and $g+c+2h\geqslant 0$.

Subcase 1.1.2.1.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g>0$; $2h(a+b)-cg\geqslant 0$; $g+c+2h\geqslant 0$; $2h(a+b)-g(g+c)>0$. As the point $y_{\min}^{\langle 1\rangle}\in\left(-\dfrac{a+b}{g};\, 1\right]$ and there is the double parabolic inequality (24), then with the identity (16) the minimum of the function (6)

$$\min_{y\in Y}\max_{x\in X}S(x,\, y)=\min_{y\in Y}\max_{x\in X}S(x,\, y)=\min\left\{\min_{y\in\left[0;\, -\frac{a+b}{g}\right]}S(0,\, y),\ \min_{y\in\left[-\frac{a+b}{g};\, 1\right]}S(1,\, y)\right\}=$$

$$=\min\left\{\min\left\{S(0,\, 0),\, S\left(0,\, -\frac{a+b}{g}\right)\right\},\, S\left(1,\, y_{\min}^{\langle 1\rangle}\right)\right\}=\min\left\{S\left(0,\, -\frac{a+b}{g}\right),\, S\left(1,\, y_{\min}^{\langle 1\rangle}\right)\right\}=$$

$$=S\left(1,\, y_{\min}^{\langle 1\rangle}\right)=S\left(1,\, -\frac{g+c}{2h}\right)=a+b+g\left(-\frac{g+c}{2h}\right)+c\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k=$$

$$=a+b+(g+c)\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k=a+b-\frac{(g+c)^2}{2h}+\frac{(g+c)^2}{4h}+k=a+b-\frac{(g+c)^2}{4h}+k=V_{\text{opt}} \qquad (26)$$

is reached on the set

$$\mathscr{Y}_{\text{opt}}=Y_{\text{opt}}=\left\{-\frac{g+c}{2h}\right\}=\{y_{\text{opt}}\}. \qquad (27)$$

The corresponding equation (19) is

$$V_{\text{opt}} = S\left(1,\, -\frac{g+c}{2h}\right) = a + b - \frac{(g+c)^2}{4h} + k = ax^2 + bx + gx\left(-\frac{g+c}{2h}\right) + c\left(-\frac{g+c}{2h}\right) + h\left(-\frac{g+c}{2h}\right)^2 + k =$$

$$= ax^2 + x\left[\frac{2bh - g(g+c)}{2h}\right] - c\frac{g+c}{2h} + \frac{(g+c)^2}{4h} + k = S\left(x,\, -\frac{g+c}{2h}\right) = S(x,\, y_{\text{opt}})\,; \qquad (28)$$

$$ax^2 + x\left[\frac{2bh - g(g+c)}{2h}\right] - c\frac{g+c}{2h} + \frac{(g+c)^2}{4h} - a - b + \frac{(g+c)^2}{4h} = ax^2 + x\left[\frac{2bh - g(g+c)}{2h}\right] + \frac{g(g+c) - 2h(a+b)}{2h} =$$

$$= a\left(x^2 + x\left[\frac{2bh - g(g+c)}{2ah}\right] + \frac{g(g+c) - 2h(a+b)}{2ah}\right) = a(x-1)\left(x - \frac{g(g+c) - 2h(a+b)}{2ah}\right) = 0\,. \qquad (29)$$

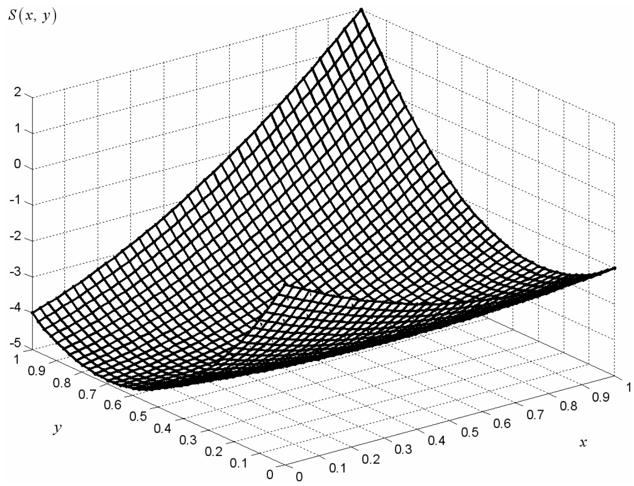

Fig. 2. The subcase $a > 0$, $b < 0$, $g > 0$, $c < 0$, $a + b < 0$, $a + b + g > 0$, $2h(a+b) - cg \geqslant 0$, $g + c + 2h \geqslant 0$, $2h(a+b) - g(g+c) > 0$ **with the kernel** $S(x, y) = 3x^2 - 5x + 8xy - 14y + 10y^2$ **and solution** $\mathscr{S} = \left\{\{1\},\, \left\{\frac{3}{10}\right\},\, -\frac{29}{10}\right\}$

It is seen from the equation (29) that the roots of the equation (28) are $x_1 = \frac{g(g+c) - 2h(a+b)}{2ah}$ and $x_2 = 1$. But $\frac{g(g+c) - 2h(a+b)}{2ah} < 0$, then $x_1 \notin X$, $x_2 \in X$ and there is the set

$$X_{\text{opt}} = \{x_2\} = \{1\} = \mathscr{X}_{\text{opt}}\,, \qquad (30)$$

whence the investigated subcase (fig. 2) game solution is

$$\mathscr{S} = \left\{\{1\},\, \left\{-\frac{g+c}{2h}\right\},\, a + b - \frac{(g+c)^2}{4h} + k\right\}. \qquad (31)$$

Subcase 1.1.2.1.2. $a > 0$, $b < 0$, $g > 0$, $c < 0$; $a + b < 0$; $a + b + g > 0$; $2h(a+b) - cg \geqslant 0$; $g + c + 2h \geqslant 0$; $2h(a+b) - g(g+c) \leqslant 0$. As the point $y_{\min}^{\langle 0\rangle} \geqslant -\frac{a+b}{g}$ then there is the double parabolic inequality (24), and as the point $y_{\min}^{\langle 1\rangle} \leqslant -\frac{a+b}{g}$ then there is the double parabolic inequality

$$S\left(1,\, y_{\min}^{\langle 1\rangle}\right) = S\left(1,\, -\frac{g+c}{2h}\right) \leqslant S\left(1,\, -\frac{a+b}{g}\right) < S(1,\, 1)\,, \qquad (32)$$

that differs from the double parabolic inequality (15) only with the strictness in the sign. Therefore with the identity (16) the minimum of the function (6)

$$\min_{y\in Y}\max_{x\in X} S(x,\, y) = \min\left\{\min_{y\in\left[0;\, -\frac{a+b}{g}\right]} S(0,\, y),\ \min_{y\in\left[-\frac{a+b}{g};\, 1\right]} S(1,\, y)\right\} =$$

$$= \min\left\{\min\left\{S(0,\, 0),\, S\left(0,\, -\frac{a+b}{g}\right)\right\},\, \min\left\{S\left(1,\, -\frac{a+b}{g}\right),\, S(1,\, 1)\right\}\right\} =$$

$$= \min\left\{S\left(0,\, -\frac{a+b}{g}\right),\, S\left(1,\, -\frac{a+b}{g}\right)\right\} = S\left(0,\, -\frac{a+b}{g}\right) = S\left(1,\, -\frac{a+b}{g}\right) = h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = V_{\text{opt}} \qquad (33)$$

is reached on the set

$$\mathscr{Y}_{\text{opt}} = Y_{\text{opt}} = \left\{-\frac{a+b}{g}\right\} = \{y_{\text{opt}}\}\,. \qquad (34)$$

The roots of the corresponding equation (19)

$$V_{\text{opt}} = S\left(1, -\frac{a+b}{g}\right) = S\left(0, -\frac{a+b}{g}\right) = h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = ax^2 + bx + gx\left(-\frac{a+b}{g}\right) + c\left(-\frac{a+b}{g}\right) + h\left(-\frac{a+b}{g}\right)^2 +$$

$$+k = ax^2 - ax + h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = ax(x-1) + h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = S\left(x, -\frac{a+b}{g}\right) = S\left(x, y_{\text{opt}}\right) \quad (35)$$

are $x_1 = 0$ and $x_2 = 1$. They are such that $x_1 \in X$ and $x_2 \in X$, so the set

$$X_{\text{opt}} = \{x_1, x_2\} = \{0, 1\}. \quad (36)$$

May $P(x_1)$ and $P(x_2)$ be the probabilities of the first player selecting its pure strategies $x_1 = x_{\text{opt}}^{\langle 1 \rangle}$ and $x_2 = x_{\text{opt}}^{\langle 2 \rangle}$. Then the set $\mathscr{X}_{\text{opt}} = \left\{X_{\text{opt}}, \left\{P\left(x_{\text{opt}}^{\langle 1 \rangle}\right), P\left(x_{\text{opt}}^{\langle 2 \rangle}\right)\right\}\right\}$ and there are $X_{\text{opt}} = \left\{x_{\text{opt}}^{\langle 1 \rangle}, x_{\text{opt}}^{\langle 2 \rangle}\right\}$, $P\left(x_{\text{opt}}^{\langle 1 \rangle}\right) + P\left(x_{\text{opt}}^{\langle 2 \rangle}\right) = 1$. Those probabilities satisfy the double inequality [1 — 3, 5, 6]

$$S\left(x^{\langle 1 \rangle}, y_{\text{opt}}\right)P\left(x^{\langle 1 \rangle}\right) + S\left(x^{\langle 2 \rangle}, y_{\text{opt}}\right)P\left(x^{\langle 2 \rangle}\right) \leqslant V_{\text{opt}} \leqslant S\left(x_{\text{opt}}^{\langle 1 \rangle}, y\right)P\left(x_{\text{opt}}^{\langle 1 \rangle}\right) + S\left(x_{\text{opt}}^{\langle 2 \rangle}, y\right)P\left(x_{\text{opt}}^{\langle 2 \rangle}\right), \quad (37)$$

where $y \neq y_{\text{opt}}$, and $x^{\langle 1 \rangle} \neq x_{\text{opt}}^{\langle 1 \rangle}$, or $x^{\langle 2 \rangle} \neq x_{\text{opt}}^{\langle 2 \rangle}$, or $P\left(x^{\langle 1 \rangle}\right) \neq P\left(x_{\text{opt}}^{\langle 1 \rangle}\right)$. In the being investigated subcase the probabilities $P(x_1) = P(0)$ and $P(x_2) = P(1)$ may be determined from the right inequality (37):

$$V_{\text{opt}} = S\left(0, -\frac{a+b}{g}\right) = S\left(1, -\frac{a+b}{g}\right) = h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k \leqslant S(0, y)P(0) + S(1, y)P(1) =$$

$$= \left(cy + hy^2 + k\right)P(0) + \left(a + b + gy + cy + hy^2 + k\right)P(1) = cy + hy^2 + (a + b + gy)P(1) + k; \quad (38)$$

$$h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} - cy - hy^2 = h\frac{(a+b)^2 - g^2y^2}{g^2} - c\frac{a+b+gy}{g} =$$

$$= h\frac{(a+b+gy)(a+b-gy)}{g^2} - c\frac{a+b+gy}{g} = (a+b+gy)\left[\frac{h(a+b-gy) - cg}{g^2}\right] \leqslant (a+b+gy)P(1). \quad (39)$$

While $a + b + gy > 0$ then $y > -\dfrac{a+b}{g}$ and step by step $-gy < a + b$, $a + b - gy < 2(a+b)$,

$$\frac{h(a+b-gy) - cg}{g^2} < \frac{2h(a+b) - cg}{g^2}, \quad (40)$$

where from the conditions $2h(a+b) - cg \geqslant 0$ and $2h(a+b) - g(g+c) \leqslant 0$ follows that the point $\dfrac{2h(a+b)-cg}{g^2} \in [0; 1]$. And then from the inequality (39) there is an inequality for the probability $P(1)$ while $a + b + gy > 0$:

$$\frac{h(a+b-gy) - cg}{g^2} \leqslant P(1) \in \lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left[\frac{2h(a+b) - cg}{g^2} - \varepsilon; 1\right]. \quad (41)$$

While $a + b + gy < 0$ then $y < -\dfrac{a+b}{g}$ and again $-gy > a + b$, $a + b - gy > 2(a+b)$,

$$\frac{h(a+b-gy) - cg}{g^2} > \frac{2h(a+b) - cg}{g^2}, \quad (42)$$

whence from the inequality (39) there is an inequality for the probability $P(1)$ while $a + b + gy < 0$:

$$\frac{h(a+b-gy) - cg}{g^2} \geqslant P(1) \in \lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left[0; \frac{2h(a+b) - cg}{g^2} + \varepsilon\right]. \quad (43)$$

Therefore the probability $P(1)$ is the intersection of the segments in the formulas (41) and (43):

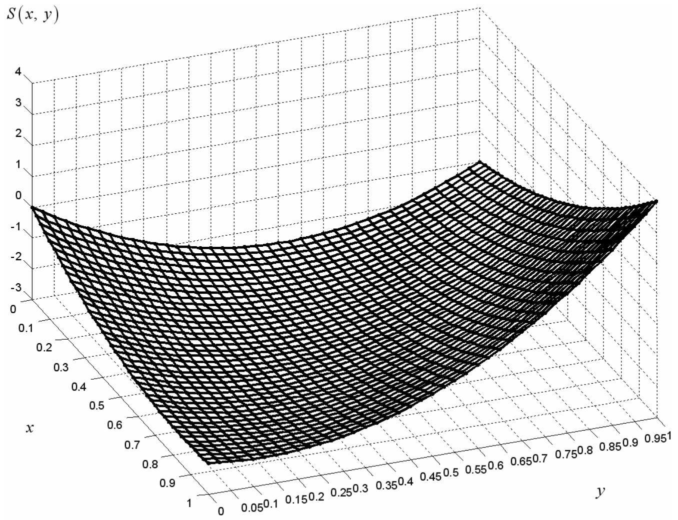

Fig. 3. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g>0$, $2h(a+b)-cg \geqslant 0$, $g+c+2h \geqslant 0$, $2h(a+b)-g(g+c) \leqslant 0$ **with the kernel** $S(x, y)=3x^2-5x+7xy-9y+8y^2$ **and solution** $\mathscr{S}=\left\{\left\{\{0, 1\}, \left\{\frac{18}{49}, \frac{31}{49}\right\}\right\}, \left\{\frac{2}{7}\right\}, -\frac{94}{49}\right\}$

$$P(1) \in \lim_{\substack{\varepsilon>0 \\ \varepsilon \to 0}} \left\{ \left[0; \frac{2h(a+b)-cg}{g^2}+\varepsilon\right] \cap \right.$$
$$\left. \cap \left[\frac{2h(a+b)-cg}{g^2}-\varepsilon; 1\right] \right\} =$$
$$= \left\{\frac{2h(a+b)-cg}{g^2}\right\}. \quad (44)$$

Hence the probability of the first player selecting its pure strategy $x_1=0$ is

$$P(0)=1-P(1)=$$
$$=1-\frac{2h(a+b)-cg}{g^2}=$$
$$=\frac{g(g+c)-2h(a+b)}{g^2}. \quad (45)$$

So in the investigated subcase (fig. 3) the set

$$\mathscr{X}_{\text{opt}}=\left\{\{0, 1\}, \left\{\frac{g(g+c)-2h(a+b)}{g^2}, \frac{2h(a+b)-cg}{g^2}\right\}\right\} \quad (46)$$

and finally the game solution is the set

$$\mathscr{S}=\left\{\left\{\{0, 1\}, \left\{\frac{g(g+c)-2h(a+b)}{g^2}, \frac{2h(a+b)-cg}{g^2}\right\}\right\}, \left\{-\frac{a+b}{g}\right\}, h\frac{(a+b)^2}{g^2}-c\frac{a+b}{g}+k\right\}. \quad (47)$$

Subcase 1.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g<0$. As here $-\frac{a+b}{g}>1$ then $a+b+gy<0$ $\forall\ y \in [0; 1]$ and the maximum of the surface (2) on the unit segment X of the variable x is the parabola

$$\max_{x \in X} S(x, y)=\max\{S(0, y), S(1, y)\}=\max\{cy+hy^2+k, a+b+gy+cy+hy^2+k\}=S(0, y)=cy+hy^2+k. \quad (48)$$

To minimize the parabola (48) on the unit segment Y, it should be determined whether $y_{\min}^{\langle 0 \rangle} \in [0; 1]$ or not. The point $-\frac{c}{2h}>0$ and $-\frac{c}{2h} \in (0; 1]$ if $c+2h \geqslant 0$; by $c+2h<0$ the point $-\frac{c}{2h}>1$.

Subcase 1.2.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g<0$; $c+2h \geqslant 0$. As the point $y_{\min}^{\langle 0 \rangle} \in (0; 1]$ then the minimum of the parabola (48)

$$\min_{y \in Y} \max_{x \in X} S(x, y)=\min_{y \in [0; 1]} S(0, y)=\min_{y \in [0; 1]} (cy+hy^2+k)=S\left(0, -\frac{c}{2h}\right)=S\left(0, y_{\min}^{\langle 0 \rangle}\right)=$$
$$=c\left(-\frac{c}{2h}\right)+h\left(-\frac{c}{2h}\right)^2+k=k-\frac{c^2}{4h}=V_{\text{opt}} \quad (49)$$

on the unit segment Y is reached on the set (18). The roots of the corresponding equation (19), which now turns into the equation (20), are $x_1=0$ and $x_2=\frac{cg-2hb}{2ah}$. But $-\frac{a+b}{g}>1$ and $-\frac{c}{2h} \in (0; 1]$, so $-\frac{c}{2h}<-\frac{a+b}{g}$ and there follows the condition $2h(a+b)-cg<0$, which means that $x_2=\frac{cg-2hb}{2ah}>1$. So those roots are $x_1 \in X$ and $x_2 \notin X$, whereupon there is the set (22) and the investigated subcase (fig. 4) game solution is the set (23).

Subcase 1.2.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g<0$; $c+2h<0$. As the point $y_{\min}^{\langle 0 \rangle}>1$ then here is the double parabolic inequality

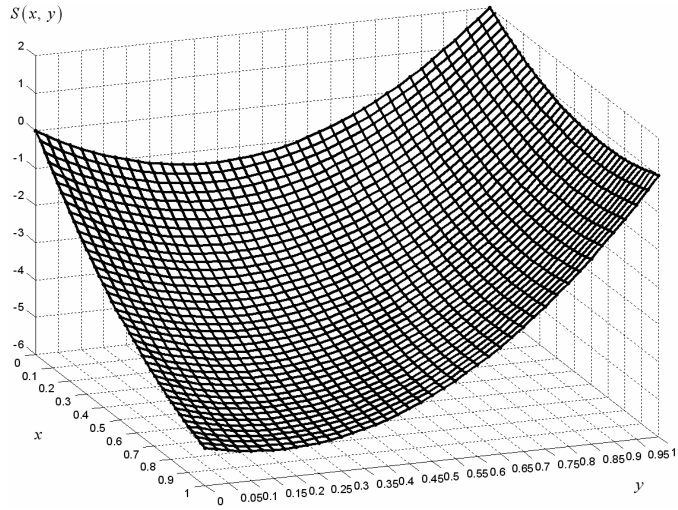

Fig. 4. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g<0$, $c+2h\geqslant 0$ **with the kernel** $S(x, y)=3x^2-8x+4xy-7y+9y^2$ **and solution** $\mathscr{S}=\left\{\{0\}, \left\{\frac{7}{18}\right\}, -\frac{49}{36}\right\}$

$$S(0, 0)>S(0, 1)> \\ >S\left(0, -\frac{c}{2h}\right)=S\left(0, y_{\min}^{\langle 0\rangle}\right), \quad (50)$$

that drives to that the minimum of the parabola (48)

$$\min_{y\in Y}\max_{x\in X} S(x, y)= \\ =\min_{y\in[0;\,1]} S(0, y)= \\ =\min_{y\in[0;\,1]}\left(cy+hy^2+k\right)= \\ =\min\left\{S(0, 0), S(0, 1)\right\}= \\ =S(0, 1)= \\ =c+h+k=V_{\text{opt}} \quad (51)$$

is reached on the set

$$\mathscr{Y}_{\text{opt}}=Y_{\text{opt}}=\{1\}=\left\{y_{\text{opt}}\right\}. \quad (52)$$

The roots of the corresponding equation (19)

$$V_{\text{opt}}=S(0, 1)=c+h+k=ax^2+bx+gx+c+h+k=ax\left(x+\frac{b+g}{a}\right)+c+h+k=S(x, 1)=S\left(x, y_{\text{opt}}\right) \quad (53)$$

are $x_1=0$ and $x_2=-\frac{b+g}{a}$. But $a+b+g<0$ means $-(b+g)>a>0$ and $-\frac{b+g}{a}>1$. Then $x_1\in X$, $x_2\notin X$ and there is the set (22), whence the investigated subcase (fig. 5) game solution is

$$\mathscr{S}=\left\{\{0\}, \{1\}, c+h+k\right\}. \quad (54)$$

Subcase 1.3. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g=0$. Here the point $-\frac{a+b}{g}=1$ and the maximum of the surface (2) on the unit segment X of the variable x is the function

$$\max_{x\in X} S(x, y)=\begin{cases} S(0, y)=cy+hy^2+k,\ y\in[0; 1), \\ S(1, y)=cy+hy^2+k=S(0, y),\ y\in\{1\}. \end{cases} \quad (55)$$

It is absolutely clear that for minimizing the function (55), that actually is the parabola, on the unit segment Y, it should be determined whether $y_{\min}^{\langle 0\rangle}\in[0; 1]$ or not. So here the point $-\frac{c}{2h}\in(0; 1)$ if $c+2h>0$ and the point $-\frac{c}{2h}\geqslant 1$ by $c+2h\leqslant 0$.

Subcase 1.3.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g=0$; $c+2h>0$. As the point $y_{\min}^{\langle 0\rangle}\in(0; 1)$ then the minimum of the parabola (55)

$$\min_{y\in Y}\max_{x\in X} S(x, y)=\min\left\{\min_{y\in[0;\,1)} S(0, y), \min_{y\in\{1\}} S(1, y)\right\}=\min\left\{S\left(0, y_{\min}^{\langle 0\rangle}\right), S(1, 1)\right\}=\min\left\{S\left(0, -\frac{c}{2h}\right), S(1, 1)\right\}=$$

$$=\min\left\{S\left(0, -\frac{c}{2h}\right), S(0, 1)\right\}=S\left(0, -\frac{c}{2h}\right)=S\left(0, y_{\min}^{\langle 0\rangle}\right)=c\left(-\frac{c}{2h}\right)+h\left(-\frac{c}{2h}\right)^2+k=k-\frac{c^2}{4h}=V_{\text{opt}} \quad (56)$$

on the unit segment Y is reached on the set (18). The roots of the corresponding equation (19), which now turns into the equation (20), are $x_1=0$ and $x_2=\frac{cg-2hb}{2ah}$. Here $-\frac{a+b}{g}=1$ and $-\frac{c}{2h}\in(0; 1)$, so again $-\frac{c}{2h}<-\frac{a+b}{g}$ and there follows the condition $2h(a+b)-cg<0$, which means that $x_2=\frac{cg-2hb}{2ah}>1$. So those roots are $x_1\in X$ and $x_2\notin X$, whereupon there is the set (22) and the investigated subcase (fig. 6) game solution is the set (23).

Subcase 1.3.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g=0$; $c+2h\leqslant 0$. As the point $y_{\min}^{\langle 0\rangle}\geqslant 1$

then here is the double parabolic inequality

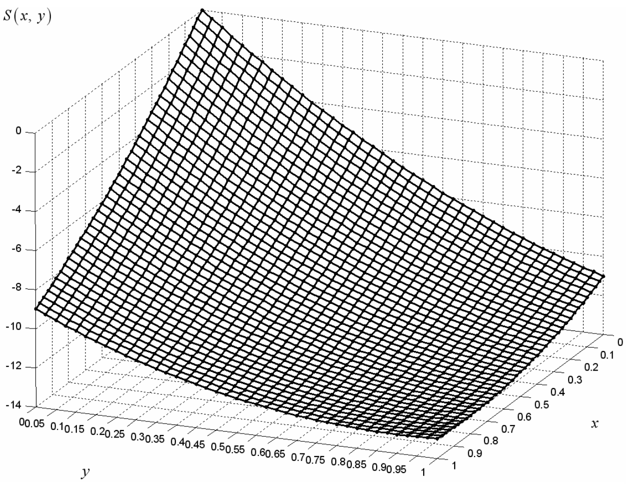

Fig. 5. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g<0$, $c+2h<0$ **with the kernel** $S(x,\,y)=3x^2-12x+7xy-16y+5y^2$ **and solution** $\mathscr{S}=\{\{0\},\,\{1\},\,-11\}$

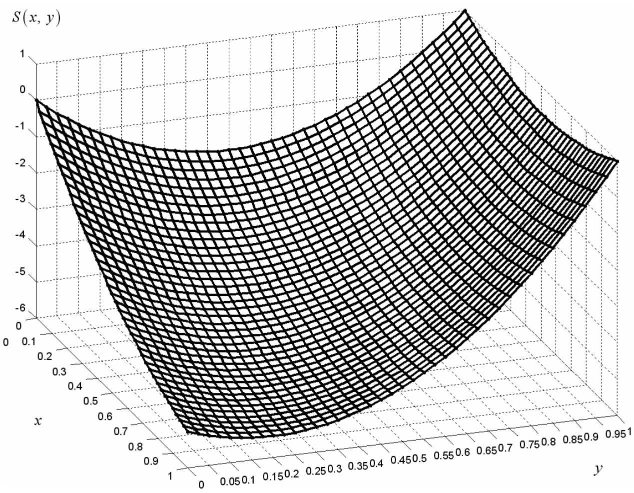

Fig. 6. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g=0$, $c+2h>0$ **with the kernel** $S(x,\,y)=3x^2-8x+5xy-9y+10y^2$ **and solution** $\mathscr{S}=\left\{\{0\},\,\left\{\frac{9}{20}\right\},\,-\frac{81}{40}\right\}$

$$S(0,\,0)>S(0,\,1)\geqslant S\left(0,\,-\frac{c}{2h}\right)=S\left(0,\,y_{\min}^{\langle 0\rangle}\right), \tag{57}$$

which with the obvious identity $S(0,\,1)=S(1,\,1)$ drives to that the minimum of the parabola (55)

$$\min_{y\in Y}\max_{x\in X}S(x,\,y)=\min\left\{\min_{y\in[0;\,1)}S(0,\,y),\,\min_{y\in\{1\}}S(1,\,y)\right\}=\min\left\{\inf_{y\in[0;\,1)}S(0,\,y),\,\min_{y\in\{1\}}S(1,\,y)\right\}=$$
$$=\min\left\{\min\left\{S(0,\,0),\,S(0,\,1)\right\},\,S(1,\,1)\right\}=\min\left\{S(0,\,1),\,S(1,\,1)\right\}=S(0,\,1)=S(1,\,1)=c+h+k=V_{\text{opt}} \tag{58}$$

is reached on the set (52). The roots of the corresponding equation (19), that is the equation (53), are $x_1=0$ and $x_2=-\frac{b+g}{a}=1$. So, $x_1\in X$, $x_2\in X$ and here is the set (36). In the being investigated subcase the corresponding probabilities $P(x_1)=P(0)$ and $P(x_2)=P(1)$ may be determined from the right inequality (37), where $y\neq y_{\text{opt}}=1$, that is $\forall\ y<1$:

$$V_{\text{opt}}=S(0,\,1)=S(1,\,1)=c+h+k\leqslant S(0,\,y)P(0)+S(1,\,y)P(1)=$$
$$=\left(cy+hy^2+k\right)P(0)+\left(a+b+gy+cy+hy^2+k\right)P(1)=cy+hy^2+(a+b+gy)P(1)+k\,; \tag{59}$$

$$c+h-cy-hy^2=c(1-y)+h(1-y)(1+y)=$$
$$=(1-y)\left[c+h(1+y)\right]\leqslant(a+b+gy)P(1)=(gy-g)P(1)=g(y-1)P(1). \tag{60}$$

As $g(y-1)<0\ \ \forall\ y<1$ then thereupon is an inequality for determining the probability $P(1)\ \ \forall\ y<1$:

$$\frac{(1-y)\left[c+h(1+y)\right]}{g(y-1)}=-\frac{c+h+hy}{g}\geqslant P(1). \tag{61}$$

However $hy<h$, $c+h+hy<c+2h$ and $-\frac{c+h+hy}{g}>-\frac{c+2h}{g}\geqslant 0$. Further there is the need to determine the location of the point $-\frac{c+2h}{g}$ relatively the right end of the unit segment $[0;\,1]$. Actually $-\frac{c+2h}{g}\in[0;\,1]$ if $c+2h+g\geqslant 0$ and $-\frac{c+2h}{g}>1$ by $c+2h+g<0$.

Subcase 1.3.2.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g=0$; $c+2h\leqslant 0$; $c+2h+g\geqslant 0$. As the point $-\dfrac{c+2h}{g} \in [0;\, 1]$ then the probability $P(1)$ is

$$P(1) \in \lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} \left\{ \left[0;\, -\frac{c+2h}{g}+\varepsilon\right] \cap [0;\, 1] \right\} = \left\{ \left[0;\, -\frac{c+2h}{g}\right] \cap [0;\, 1] \right\} = \left[0;\, -\frac{c+2h}{g}\right]. \tag{62}$$

So in the investigated subcase (fig. 7) the set

$$\mathscr{X}_{\text{opt}} = \{\{0,\, 1\},\, \{1-P(1),\, P(1)\}\} \tag{63}$$

and finally the game solution is the set

$$\mathscr{S} = \{\{\{0,\, 1\},\, \{1-P(1),\, P(1)\}\},\, \{1\},\, c+h+k\} \tag{64}$$

with the probability $P(1)$ in the formula (62).

Subcase 1.3.2.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g=0$; $c+2h \leqslant 0$; $c+2h+g<0$. As the point $-\dfrac{c+2h}{g}>1$ then the probability $P(1)$ is

$$P(1) \in \lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} \left\{ \left[0;\, -\frac{c+2h}{g}+\varepsilon\right] \cap [0;\, 1] \right\} = \left\{ \left[0;\, -\frac{c+2h}{g}\right] \cap [0;\, 1] \right\} = [0;\, 1]. \tag{65}$$

So in the investigated subcase there is the set (63) with the probability $P(1)$ in the formula (65), and the game solution (fig. 8) is the set (64).

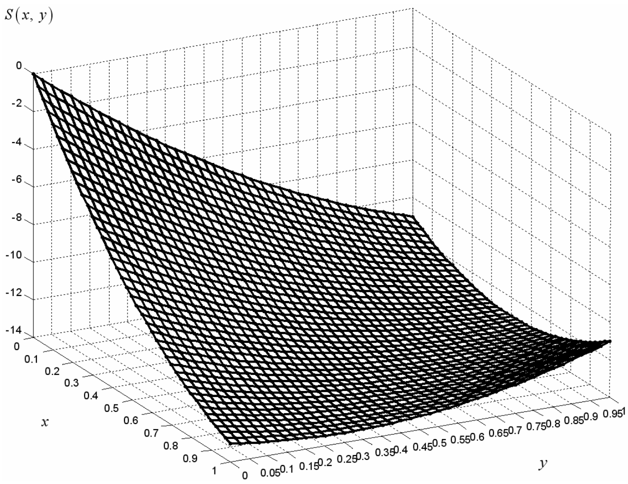

Fig. 7. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g=0$, $c+2h \leqslant 0$, $c+2h+g \geqslant 0$ **with the kernel** $S(x,\, y)=8x^2-21x+13xy-16y+5y^2$ **and solution** $\mathscr{S} = \{\{\{0,\, 1\},\, \{1-P(1),\, P(1)\}\},\, \{1\},\, -11\}$**, where** $P(1) \in \left[0;\, \frac{6}{13}\right]$

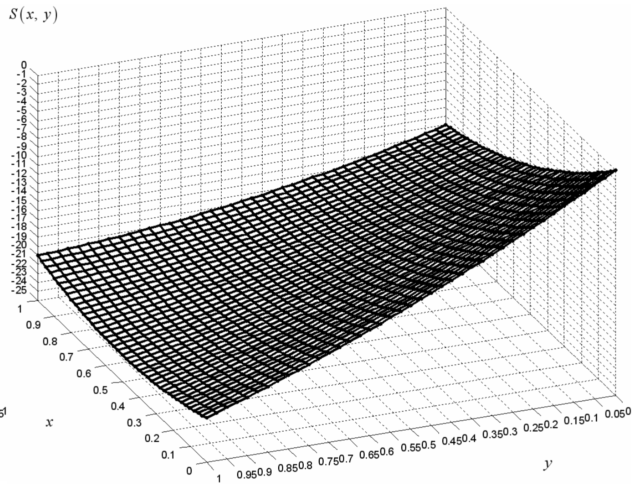

Fig. 8. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b<0$, $a+b+g=0$, $c+2h \leqslant 0$, $c+2h+g<0$ **with the kernel** $S(x,\, y)=8x^2-21x+13xy-25y+5y^2$ **and solution** $\mathscr{S} = \{\{\{0,\, 1\},\, \{1-P(1),\, P(1)\}\},\, \{1\},\, -20\}$**, where** $P(1) \in [0;\, 1]$

Subcase 2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b>0$. As the point $-\dfrac{a+b}{g}<0$ then $a+b+gy>0$ $\forall\, y \in [0;\, 1]$ and the maximum of the surface (2) on the segment X of the variable x is the parabola

$$\max_{x\in X} S(x,\, y) = \max\{S(0,\, y),\, S(1,\, y)\} = \max\{cy+hy^2+k,\, a+b+gy+cy+hy^2+k\} =$$
$$= S(1,\, y) = a+b+gy+cy+hy^2+k\,. \tag{66}$$

The minimum of the parabola (66) on the segment Y depends upon whether $y_{\min}^{\langle 1 \rangle} \in [0;\, 1]$ or $y_{\min}^{\langle 1 \rangle} \notin [0;\, 1]$. The point $y_{\min}^{\langle 1 \rangle} \in [0;\, 1]$ if $g+c \leqslant 0$ and $g+c+2h \geqslant 0$.

Subcase 2.1.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b>0$; $g+c \leqslant 0$; $g+c+2h \geqslant 0$. As the point $y_{\min}^{\langle 1 \rangle} \in [0;\, 1]$

then the minimum of the parabola (66)

$$\min_{y\in Y}\max_{x\in X} S(x, y) = \min_{y\in Y} S(1, y) = S\left(1, y_{\min}^{\langle 1\rangle}\right) = S\left(1, -\frac{g+c}{2h}\right) = a+b+g\left(-\frac{g+c}{2h}\right)+c\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k =$$
$$= a+b+(g+c)\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k = a+b-\frac{(g+c)^2}{2h}+\frac{(g+c)^2}{4h}+k = a+b-\frac{(g+c)^2}{4h}+k = V_{\text{opt}} \quad (67)$$

is reached on the set (27). The roots of the corresponding equation (19), which now turns into the equations (28) and (29), are $x_1 = \frac{g(g+c)-2h(a+b)}{2ah}$ and $x_2 = 1$. But the conditions $a+b>0$ and $g+c \leqslant 0$ give that $\frac{g(g+c)-2h(a+b)}{2ah} < 0$. So, $x_1 \notin X$, $x_2 \in X$, and there is the set (30), whence the investigated subcase (fig. 9) game solution is the set (31).

Subcase 2.1.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b>0$; $g+c \leqslant 0$; $g+c+2h<0$. Here yet the point $y_{\min}^{\langle 1\rangle} = -\frac{g+c}{2h} > 1$ and so there is the double parabolic inequality

$$S(1, 0) > S(1, 1) > S\left(1, -\frac{g+c}{2h}\right) = S\left(1, y_{\min}^{\langle 1\rangle}\right), \quad (68)$$

that drives to that the minimum of the parabola (66)

$$\min_{y\in Y}\max_{x\in X} S(x, y) = \min_{y\in[0;\,1]} S(1, y) = \min_{y\in[0;\,1]}\left(a+b+gy+cy+hy^2+k\right) =$$
$$= \min\left\{S(1, 0), S(1, 1)\right\} = S(1, 1) = a+b+g+c+h+k = V_{\text{opt}} \quad (69)$$

is reached on the set (52). The roots of the corresponding equation (19)

$$V_{\text{opt}} = S(1, 1) = a+b+g+c+h+k = ax^2+bx+gx+c+h+k =$$
$$= a(x-1)\left(x+\frac{a+b+g}{a}\right)+a+b+g+c+h+k = S(x, 1) = S\left(x, y_{\text{opt}}\right) \quad (70)$$

are $x_1 = -\frac{a+b+g}{a}$ and $x_2 = 1$. But if $a+b>0$ then $a+b+g>0$ and $-\frac{a+b+g}{a}<0$. So $x_1 \notin X$, $x_2 \in X$ and there is the set (30), whence the investigated subcase (fig. 10) game solution is

$$\mathscr{S} = \left\{\{1\}, \{1\}, a+b+g+c+h+k\right\}, \quad (71)$$

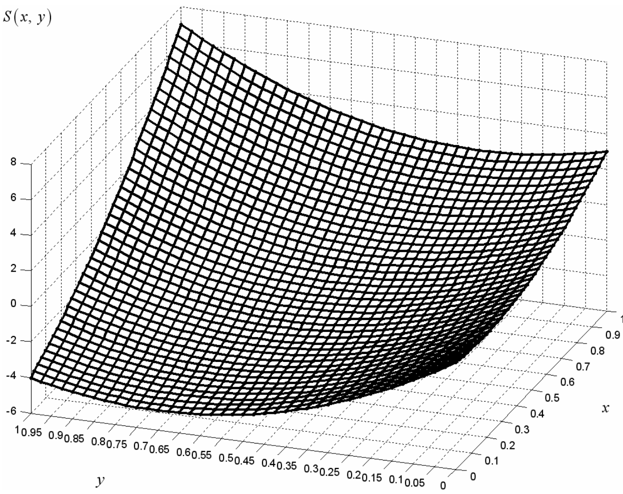

Fig. 9. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b>0$, $g+c \leqslant 0$, $g+c+2h \geqslant 0$ **with the kernel** $S(x, y) = 5x^2-2x+8xy-14y+10y^2$ **and solution** $\mathscr{S} = \left\{\{1\}, \left\{\frac{3}{10}\right\}, \frac{21}{10}\right\}$

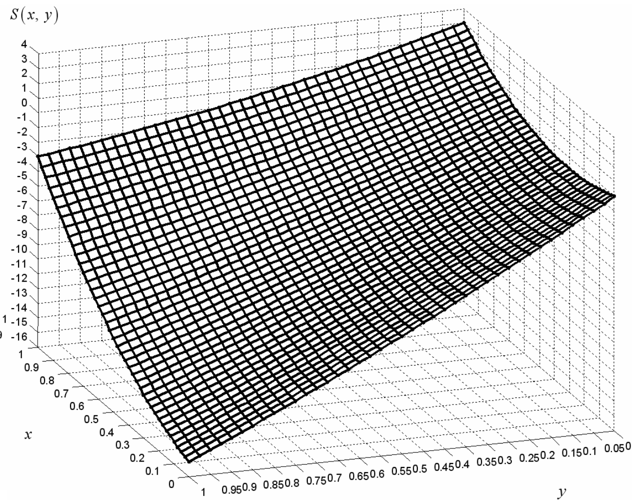

Fig. 10. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b>0$, $g+c \leqslant 0$, $g+c+2h<0$ **with the kernel** $S(x, y) = 8x^2-5x+9xy-17y+2y^2$ **and solution** $\mathscr{S} = \left\{\{1\}, \{1\}, -3\right\}$

being the same as for the intentionally missed **subcase 1.1.2.2.** $a>0$, $b<0$, $g>0$, $c<0$; $a+b<0$; $a+b+g>0$; $2h(a+b)-cg \geqslant 0$; $g+c+2h<0$.

Subcase 2.2. $a>0$, $b<0$, $g>0$, $c<0$; $a+b>0$; $g+c>0$. The point $y_{\min}^{\langle 1\rangle}=-\dfrac{g+c}{2h}<0$ and there is the corresponding double parabolic inequality

$$S\left(1,\, y_{\min}^{\langle 1\rangle}\right)=S\left(1,\, -\frac{g+c}{2h}\right)<S(1,\,0)<S(1,\,1). \tag{72}$$

This inequality gives that the minimum of the parabola (66)

$$\min_{y\in Y}\max_{x\in X} S(x,\,y)=\min_{y\in[0;\,1]} S(1,\,y)=\min_{y\in[0;\,1]}\left(a+b+gy+cy+hy^2+k\right)=$$
$$=\min\left\{S(1,\,0),\, S(1,\,1)\right\}=S(1,\,0)=a+b+k=V_{\text{opt}} \tag{73}$$

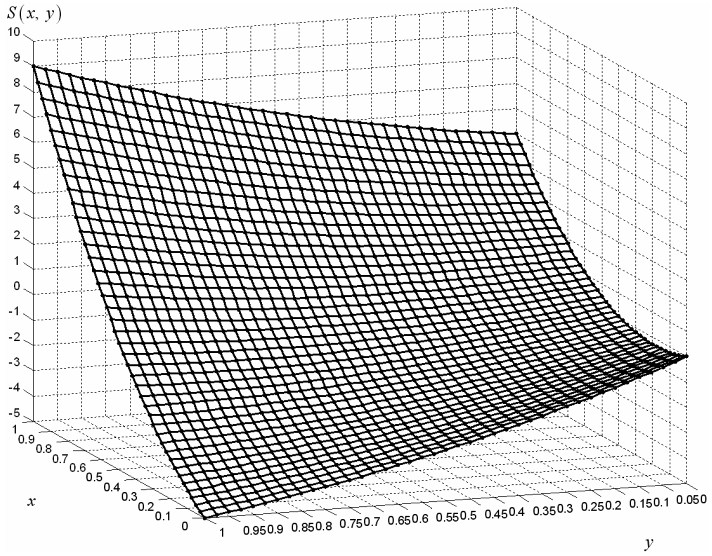

Fig. 11. The subcase $a>0$, $b<0$, $g>0$, $c<0$, $a+b>0$, $g+c>0$ with the kernel $S(x,\,y)=8x^2-3x+9xy-7y+2y^2$ and solution $\mathscr{S}=\{\{1\},\,\{0\},\,5\}$

is reached on the set

$$\mathscr{Y}_{\text{opt}}=Y_{\text{opt}}=\{0\}=\left\{y_{\text{opt}}\right\}. \tag{74}$$

In the subcase being investigated the roots of the corresponding equation (19)

$$V_{\text{opt}}=S(1,\,0)=a+b+k=$$
$$=ax^2+bx+k=$$
$$=a(x-1)\left(x+\frac{a+b}{a}\right)+a+b+k=$$
$$=S(x,\,0)=S\left(x,\,y_{\text{opt}}\right) \tag{75}$$

are $x_1=-\dfrac{a+b}{a}$ and $x_2=1$. But $-\dfrac{a+b}{a}<0$ and $x_1\notin X$, $x_2\in X$, so there is the set (30), and accordingly the investigated subcase (fig. 11) game solution is the set

$$\mathscr{S}=\left\{\{1\},\,\{0\},\,a+b+k\right\}. \tag{76}$$

Subcase 3. $a>0$, $b<0$, $g>0$, $c<0$; $a+b=0$. Here the point $-\dfrac{a+b}{g}=0$ and the maximum of the surface (2) on the unit segment X of the variable x is the function

$$\max_{x\in X} S(x,\,y)=\begin{cases} S(1,\,y)=a+b+gy+cy+hy^2+k=gy+cy+hy^2+k,\ y\in(0;\,1], \\ S(0,\,y)=cy+hy^2+k=S(1,\,y),\ y\in\{0\}. \end{cases} \tag{77}$$

It is absolutely clear that for minimizing the function (77), that actually is the parabola, on the unit segment Y, it should be determined whether $y_{\min}^{\langle 1\rangle}\in[0;\,1]$ or not. So here the point $-\dfrac{g+c}{2h}\in(0;\,1]$ if $g+c<0$ and $g+c+2h\geqslant 0$.

Subcase 3.1.1. $a>0$, $b<0$, $g>0$, $c<0$; $a+b=0$; $g+c<0$; $g+c+2h\geqslant 0$. As the point $y_{\min}^{\langle 1\rangle}\in(0;\,1]$ then the minimum of the parabola (77) is the statement (67), being reached on the set (27). The roots of the corresponding equation (19), which now turns into the equations (28) and (29), are $x_1=\dfrac{g(g+c)-2h(a+b)}{2ah}$ and $x_2=1$. But the conditions $a+b=0$ and $g+c<0$ give that $x_1=\dfrac{g(g+c)}{2ah}<0$. So, $x_1\notin X$, $x_2\in X$, and there is the set (30), whence the investigated subcase (fig. 12) game solution is the set (31), which in this subcase is the set

$$\mathscr{S}=\left\{\{1\},\,\left\{-\frac{g+c}{2h}\right\},\,k-\frac{(g+c)^2}{4h}\right\}. \tag{78}$$

Subcase 3.1.2. $a > 0$, $b < 0$, $g > 0$, $c < 0$; $a + b = 0$; $g + c < 0$; $g + c + 2h < 0$. Here yet the point $y_{\min}^{\langle 1 \rangle} = -\dfrac{g+c}{2h} > 1$ and there is the double parabolic inequality (68), that drives to that the minimum of the parabola (77)

$$\min_{y \in Y} \max_{x \in X} S(x,\, y) = \min\left\{ \min_{y \in (0;\, 1]} S(1,\, y),\, \min_{y \in \{0\}} S(0,\, y) \right\} = \min\left\{ \min\left\{ S(1,\, 0),\, S(1,\, 1) \right\},\, S(0,\, 0) \right\} =$$
$$= \min\left\{ S(1,\, 1),\, S(0,\, 0) \right\} = \min\left\{ S(1,\, 1),\, S(1,\, 0) \right\} = S(1,\, 1) = a + b + g + c + h + k = g + c + h + k = V_{\text{opt}} \qquad (79)$$

is reached on the set (52). The roots of the corresponding equation (19), that now from the equation (70) turns to the equation

$$V_{\text{opt}} = S(1,\, 1) = a + b + g + c + h + k = g + c + h + k = ax^2 + bx + gx + c + h + k =$$
$$= a(x-1)\left(x + \frac{a+b+g}{a} \right) + a + b + g + c + h + k = a(x-1)\left(x + \frac{g}{a} \right) + g + c + h + k = S(x,\, 1) = S(x,\, y_{\text{opt}}), \qquad (80)$$

are $x_1 = -\dfrac{g}{a}$ and $x_2 = 1$. But here $-\dfrac{g}{a} < 0$, so $x_1 \notin X$, $x_2 \in X$ and there is the set (30), whence the investigated subcase (fig. 13) game solution is

$$\mathscr{S} = \left\{ \{1\},\, \{1\},\, g + c + h + k \right\}, \qquad (81)$$

though it could have been stated as the set (71).

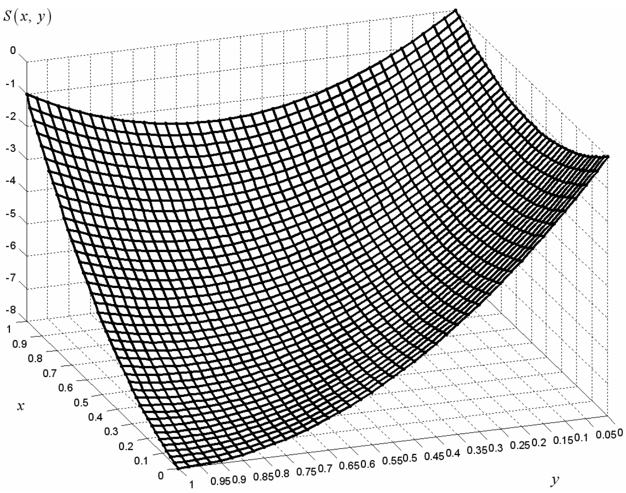

Fig. 12. The subcase $a > 0$, $b < 0$, $g > 0$, $c < 0$, $a + b = 0$, $g + c < 0$, $g + c + 2h \geqslant 0$ with the kernel $S(x,\, y) = 5x^2 - 5x + 7xy - 16y + 8y^2$ and solution $\mathscr{S} = \left\{ \{1\},\, \left\{ \dfrac{9}{16} \right\},\, -\dfrac{81}{32} \right\}$

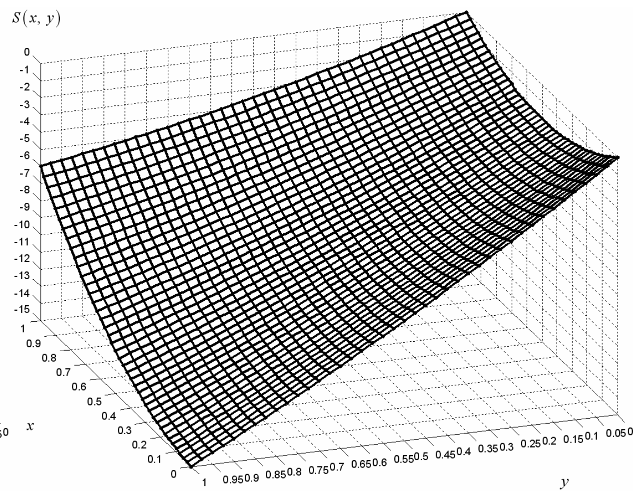

Fig. 13. The subcase $a > 0$, $b < 0$, $g > 0$, $c < 0$, $a + b = 0$, $g + c < 0$, $g + c + 2h < 0$ with the kernel $S(x,\, y) = 8x^2 - 8x + 9xy - 17y + 2y^2$ and solution $\mathscr{S} = \left\{ \{1\},\, \{1\},\, -6 \right\}$

Subcase 3.2. $a > 0$, $b < 0$, $g > 0$, $c < 0$; $a + b = 0$; $g + c \geqslant 0$. As the point $y_{\min}^{\langle 1 \rangle} = -\dfrac{g+c}{2h} \leqslant 0$ then there is the corresponding double parabolic inequality

$$S\left(1,\, y_{\min}^{\langle 1 \rangle}\right) = S\left(1,\, -\frac{g+c}{2h}\right) \leqslant S(1,\, 0) < S(1,\, 1). \qquad (82)$$

This inequality drives to that the minimum of the parabola (77)

$$\min_{y \in Y} \max_{x \in X} S(x,\, y) = \min\left\{ \min_{y \in (0;\, 1]} S(1,\, y),\, \min_{y \in \{0\}} S(0,\, y) \right\} =$$
$$= \min\left\{ \min\left\{ S(1,\, 0),\, S(1,\, 1) \right\},\, S(0,\, 0) \right\} = \min\left\{ S(1,\, 0),\, S(0,\, 0) \right\} = S(1,\, 0) = S(0,\, 0) = k = V_{\text{opt}} \qquad (83)$$

is reached on the set (74). In the subcase being investigated the roots of the corresponding equation (19)

$$V_{\text{opt}} = S(1,\,0) = S(0,\,0) = k = ax^2 + bx + k = ax^2 - ax + k = ax(x-1) + k = S(x,\,0) = S(x,\,y_{\text{opt}}) \tag{84}$$

are $x_1 = 0$ and $x_2 = 1$, so there is the set (36). In the being investigated subcase probabilities $P(x_1) = P(0)$ and $P(x_2) = P(1)$ may be determined from the right inequality (37), where $y \neq y_{\text{opt}} = 0$:

$$V_{\text{opt}} = S(0,\,0) = S(1,\,0) = k \leqslant S(0,\,y)P(0) + S(1,\,y)P(1) = (cy + hy^2 + k)P(0) +$$

$$+(a + b + gy + cy + hy^2 + k)P(1) = cy + hy^2 + (a + b + gy)P(1) + k = cy + hy^2 + gyP(1) + k\,; \tag{85}$$

$$0 \leqslant cy + hy^2 + gyP(1)\,; \tag{86}$$

$$gyP(1) \geqslant -cy - hy^2\,. \tag{87}$$

As $gy > 0$ than dividing the members of the inequality (87) by gy will not change the sign in it. Hence the probability $P(1)$ evaluation is

$$P(1) \geqslant -\frac{c + hy}{g}\,. \tag{88}$$

But inasmuch as $hy > 0$, $c + hy > c$, $c < 0$ and $g > 0$, then

$$-\frac{c + hy}{g} < -\frac{c}{g} \quad \forall\; y \in (0;\,1] \tag{89}$$

and $g + c \geqslant 0$ means that $-\frac{c}{g} \in (0;\,1]$. Therefore the probability $P(1)$ here satisfies the condition

$$P(1) \in \lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left\{ \left[-\frac{c}{g} - \varepsilon;\, 1 \right] \bigcap [0;\,1] \right\} = \left\{ \left[-\frac{c}{g};\, 1 \right] \bigcap [0;\,1] \right\} = \left[-\frac{c}{g};\, 1 \right]. \tag{90}$$

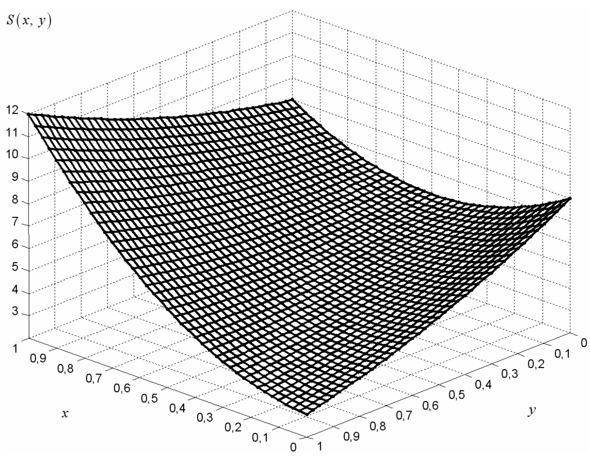

Fig. 14. The subcase $a > 0$, $b < 0$, $g > 0$, $c < 0$, $a + b = 0$, $g + c \geqslant 0$ with the kernel $S(x,\,y) = 8x^2 - 8x + 9xy - 7y + 2y^2 + 8$ and solution $\mathscr{S} = \left\{\left\{\{0,\,1\},\,\{1 - P(1),\,P(1)\}\right\},\,\{0\},\,8\right\}$, where $P(1) \in \left[\frac{7}{9};\,1\right]$

The probability $P(0)$ satisfies the condition

$$P(0) \in \left[0;\,\frac{g + c}{g}\right] \tag{91}$$

that is from that $1 - \left(-\frac{c}{g}\right) = \frac{g + c}{g}$. Resuming, in the investigated subcase (fig. 14) there is the set (63) and the game solution

$$\mathscr{S} = \left\{\left\{\{0,\,1\},\,\{1 - P(1),\,P(1)\}\right\},\,\{0\},\,k\right\}. \tag{92}$$

with the probability $P(1)$ in the formula (90).

Conclusion

Having investigated the 15 subcases of the coefficients interrelationships in the kernel (2), there have been determined the corresponding nine types of the antagonistic game solution. They are grouped in the table 1.

Table 1

The solutions of the investigated continuous strictly convex antagonistic game with the kernel

$$S(x,\,y) = ax^2 + bx + gxy + cy + hy^2 + k$$

The given game kernel attributes with $a > 0$, $b < 0$, $g > 0$, $c < 0$	The game solution $\mathscr{S} = \{\mathscr{X}_{\text{opt}},\,\mathscr{Y}_{\text{opt}},\,V_{\text{opt}}\}$
$a + b < 0$, $a + b + g > 0$, $2h(a + b) - cg < 0$	$\mathscr{S} = \left\{\{0\},\,\left\{-\frac{c}{2h}\right\},\,k - \frac{c^2}{4h}\right\}$
$a + b < 0$, $a + b + g < 0$, $c + 2h \geqslant 0$	
$a + b < 0$, $a + b + g = 0$, $c + 2h > 0$	

<table>
<tr><th>The given game kernel attributes with $a>0$, $b<0$, $g>0$, $c<0$</th><th>The game solution $\mathscr{S}=\left\{\mathscr{X}_{\text{opt}}, \mathscr{Y}_{\text{opt}}, V_{\text{opt}}\right\}$</th></tr>
<tr><td>$a+b<0$, $a+b+g>0$, $2h(a+b)-cg\geqslant 0$, $g+c+2h\geqslant 0$, $2h(a+b)-g(g+c)>0$</td><td rowspan="3">$\mathscr{S}=\left\{\{1\}, \left\{-\dfrac{g+c}{2h}\right\}, a+b-\dfrac{(g+c)^2}{4h}+k\right\}$</td></tr>
<tr><td>$a+b>0$, $g+c\leqslant 0$, $g+c+2h\geqslant 0$</td></tr>
<tr><td>$a+b=0$, $g+c<0$, $g+c+2h\geqslant 0$</td></tr>
<tr><td>$a+b<0$, $a+b+g>0$, $2h(a+b)-cg\geqslant 0$, $g+c+2h\geqslant 0$, $2h(a+b)-g(g+c)\leqslant 0$</td><td>$\mathscr{X}_{\text{opt}}=\left\{\{0, 1\}, \left\{\dfrac{g(g+c)-2h(a+b)}{g^2}, \dfrac{2h(a+b)-cg}{g^2}\right\}\right\}$,
$\mathscr{S}=\left\{\mathscr{X}_{\text{opt}}, \left\{-\dfrac{a+b}{g}\right\}, h\dfrac{(a+b)^2}{g^2}-c\dfrac{a+b}{g}+k\right\}$</td></tr>
<tr><td>$a+b<0$, $a+b+g<0$, $c+2h<0$</td><td>$\mathscr{S}=\left\{\{0\}, \{1\}, c+h+k\right\}$</td></tr>
<tr><td>$a+b<0$, $a+b+g=0$, $c+2h\leqslant 0$, $c+2h+g\geqslant 0$</td><td>$\mathscr{S}=\left\{\left\{\{0, 1\}, \{1-P(1), P(1)\}\right\}, \{1\}, c+h+k\right\}$,
$P(1)\in\left[0; -\dfrac{c+2h}{g}\right]$</td></tr>
<tr><td>$a+b<0$, $a+b+g=0$, $c+2h\leqslant 0$, $c+2h+g<0$</td><td>$\mathscr{S}=\left\{\left\{\{0, 1\}, \{1-P(1), P(1)\}\right\}, \{1\}, c+h+k\right\}$, $P(1)\in[0; 1]$</td></tr>
<tr><td>$a+b>0$, $g+c\leqslant 0$, $g+c+2h<0$</td><td rowspan="3">$\mathscr{S}=\left\{\{1\}, \{1\}, a+b+g+c+h+k\right\}$</td></tr>
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<tr><td>$a+b>0$, $g+c>0$</td><td>$\mathscr{S}=\left\{\{1\}, \{0\}, a+b+k\right\}$</td></tr>
<tr><td>$a+b=0$, $g+c\geqslant 0$</td><td>$\mathscr{S}=\left\{\left\{\{0, 1\}, \{1-P(1), P(1)\}\right\}, \{0\}, k\right\}$, $P(1)\in\left[-\dfrac{c}{g}; 1\right]$</td></tr>
</table>

References

1. Romanuke V. V. The figured 10 subcases of the coefficients interrelationships in the kernel of a continuous strictly convex antagonistic game with the corresponding six types of the solution / V. V. Romanuke // Науково-теоретичний журнал Хмельницького економічного університету “Наука й економіка”. — Випуск 2 (14), 2009. — С. 308 — 326.
2. Romanuke V. V., Nezdorovin V. P. The figured 12 subcases of the coefficients interrelationships in the kernel of a continuous strictly convex antagonistic game with the corresponding eight types of the solution / V. V. Romanuke // Збірник наукових праць факультету прикладної математики та комп’ютерних технологій Хмельницького національного університету. — 2009. — № 1 (2). — С. 93 — 105.
3. Romanuke V. V. Figuring out the four subcases of the interrelationships of the coefficients by the first player pure strategy in the kernel of a continuous strictly convex antagonistic game with the corresponding three types of the solution / V. V. Romanuke // Вісник Хмельницького національного університету. Економічні науки. — 2009. — № 5. — Т. 3. — С. 274 — 281.
4. Теория игр: Учеб. пособие для ун-тов / Л. А. Петросян, Н. А. Зенкевич, Е. А. Семина. — М.: Высшая школа, Книжный дом “Университет”, 1998. — 304 с.: ил.
5. Romanuke V. V. The convex game on the unit square with the kernel, that is the sum of the weighted strategies and their weighted product / V. V. Romanuke // Математическое моделирование, обратные задачи, информационно-вычислительные технологии: сборник статей VII Международной научно-технической конференции. Ч. II. — Пенза: РИО ПГСХА, 2007. — С. 73 — 77.
6. Romanuke V. V. Optimality control in the concave antagonistic game with annihilation probability payoff function as the kernel on the unit hypercube of the six-dimensional arithmetic space / V. V. Romanuke // Информационно-вычислительные технологии и их приложения: сборник статей X Международной научно-технической конференции. — Пенза: РИО ПГСХА, 2009. — С. 236 — 241.

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