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SOFTWARE FOR FOURIER ANALYSIS OF FUNCTIONS WITH VARIABLE PERIOD

The analysis of modern software of computer mathematics providing reasons to carry our investigations of periodic functions, i.e., functions, which period, without any reservations, is considered to be constant. However, in practice, there are functions, for example, electrocardiograms obtained after physical exercises, which period is no longer constant, and changes in some way. Several results of the investigation of periodic functions with the variable period are represented in the article: the definition of such functions, methods and examples of their analytical assignment and analytical expressions of variables are given. Taking into account the fact that the main tools for periodic functions analysis are Fourier series, the software of Fourier analysis are considered and the software for such analysis of periodic functions with a variable period is developed providing the calculation of Fourier coefficients and construction of graphs for such functions and their finite Fourier series, which, in turn, enables us to continue the complex investigation of periodic functions with variable period.

Key words: *variable period, periodic functions with variable period, Fourier coefficients of periodic functions with variable period, Fourier series of periodic functions with variable period, software systems for the analysis of periodic functions with variable period.*

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ПРОГРАМНЕ ЗАБЕЗПЕЧЕННЯ ДЛЯ ФУР'Є-АНАЛІЗУ ФУНКЦІЙ ЗІ ЗМІННИМ ПЕРІОДОМ

Проведено аналіз сучасних програмних засобів комп'ютерної математики, які дозволяють проводити дослідження періодичних функцій, тобто функцій, період яких без будь-яких застережень вважається постійним. Однак, на практиці зустрічаються функції, для прикладу, електрокардіограми, отримані після фізичного навантаження, період яких вже не є постійним, а певним чином змінюється. В статті наведені деякі результати дослідження періодичних функцій зі змінним періодом: надано визначення таких функцій, методи та приклади їх аналітичного задання та аналітичні вирази змінних періодів. Враховуючи, що основним інструментом аналізу періодичних функцій є ряди Фур'є, розглядаються програмні засоби Фур'є-аналізу та розроблено програмне забезпечення для такого аналізу періодичних функцій зі змінним періодом, яке дозволяє розраховувати коефіцієнти Фур'є і будувати графіки таких функцій та їх скінчених рядів Фур'є, що, в свою чергу, дає можливість продовжувати комплексне вивчення періодичних функцій зі змінним періодом.

Ключові слова: змінний період, періодичні функції зі змінним періодом, коефіцієнти Фур'є періодичних функцій зі змінним періодом, ряд Фур'є періодичних функцій зі змінним періодом, програмні системи аналізу періодичних функцій зі змінним періодом.

1. The state of the problem and the relevance of the work. The main components of modern information technology are the selection of information, storage of BigData on carriers, its transmission to any distance in a short period of time, computer processing of information according to the given algorithms. It is assumed that the carriers of information are various empirical signals (electrical, optical, acoustic, electromagnetic, etc.), sometimes called functions, processes. One of the most common methods for empirical functions processing is to replace them by analytical expressions convenient for analysis tasks. The series are mainly used as the tool of replacement, i.e., the functions presentation of in the form of the sum of certain elementary functions with corresponding coefficients. In cases when trigonometric sinus and cosine functions are chosen as elementary functions, the resulting series are called Fourier series, and the coefficients of the series are Fourier coefficients. Knowing the Fourier coefficients provides solving the problems of their spectral analysis, filtering (selection of the useful signal against obstacles), choosing the sampling frequency of the continuous function, etc. Although the periodic functions have been thoroughly investigated, there are periodic functions, which period is no longer constant, but changes in some way. The most indicative examples of such functions are electrocardiograms obtained during or after organism stimulation by certain exciter, for example, physical activity. A number of important steps have already been made in order to investigate the periodic empirical functions with variable period (PFVP). In paper [1], the definition of PFVP is given and some properties of the variable period are considered. For the case when the variable period is unknown, the development of its estimation method is described [2], [3]. Paper [4] is devoted to the problems of creation of orthogonal trigonometric functions with variable period and determination of their variable period. The availability of the class of functions with variable period [5] and corresponding orthogonal systems opens the way to the solution of the problem of approximation of functions with variable period. Therefore, the problem of information technologies development provides the construction of Fourier series of PFVP, the calculation of Fourier coefficients, and the calculation of the spectra of such functions is important.

The objective of this paper is to analyze the modern mathematical programming environments of Fourier analysis of periodic functions and to create software for the Fourier analysis of periodic functions with variable period, particularly the determination of their Fourier coefficients and the construction of Fourier series.

2. Mathematical software systems for information investigations. At present, there is a large number of software tools for computer mathematics. The classification of software for computer mathematics [6] is shown in Fig. 1.

Among the software tools of computer mathematics there are universal mathematical systems (Derive, Maple, MuPAD, Mathematica, Mathcad, Mathcad, MATLAB) which make possible to solve scientific, engineering and educational problems, particularly the problems of spectral analysis and synthesis by numerical methods.

For example, in the Derive mathematical package there is the function for the function expansion in trigonometric Fourier series with syntax: *FOURIER(y,t,t1,t2,n)* – represents the expression in the form of the sum with *n* harmonics of the trigonometric Fourier series, approximating function *y(t)* in the interval *t* from $t=t_1$ to $t=t_2$.

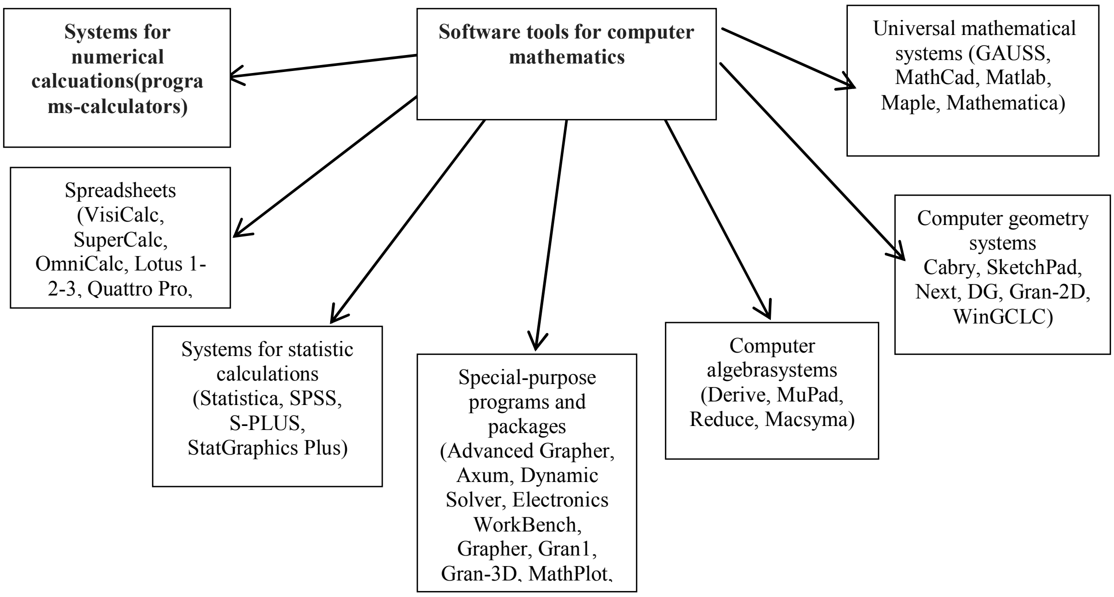

Fig. 1. Classification of computer mathematics software

There are two functions of Fast Fourier Transform (FFT) in: *fft(list,n)* and *ifft(list,n)* – direct and inverse Fourier transform over vector-list with 2n elements relatively.

Maple has two functions for calculating cosine and sinus Fourier integrals for the function f(t): *fouriercos(expr,t,s), fouriersin(expr,t,s).*

Mathematica system has built-in functions: *Fourier[list], InverseFourier[list]*, performing direct and inverse Fourier transform of *list*, using FFT algorithm. In addition, Mathematica uses functions for calculating Fourier series coefficients, called cosine and sinus Fourier integrals: *FourierSinTransform, InverseFourierSinTransform*. For more complete Fourier transform, there are extended functions included in Fourier Transform subpacket of Calculus package.

One of the systems having the greatest potential for numerical spectral analysis is Mathcad, the main functions are: *fft(v)* та *ifft(v)* – performs relatively FFT and inverse Fourier transform for vector v with complex numbers. Mathcad has a large set of tools for solving spectral analysis problems and a user-friendly interface. However, the low data processing speed is Mathcad disadvantage.

MATLAB [7] computer mathematics system has the following functions: *fft (X)* -discrete Fourier transform *(DFT), fft (X, n)* - n-point Fourier transforms. The advantage of this package is the calculation speed. The disadvantages include certain inconvenience and requirements to PC resources.

In modern computer-aided design programs (ECAD), fast Fourier transform is used in order to carry out spectral analysis. However, since the main condition for the Fourier analysis application is the signal frequency, errors are generated in calculations, and obtained spectral characteristics is inadequate [8] because of the availability of the nonperiodic transient process.

Besides universal mathematical systems there are online calculators on the Internet network allowing to conduct Fourier analysis. In WolframAlpha [9] we can use *Fourier transform* function resulting in the character result. The use of the *Fourier series* function, which makes it possible to display the graphical representation of the expansion on separate harmonics is also predicted. Online-calculator Symbolab [10] has the *Fourier series* function. The peculiarity of the web-complement MathsTools [11] is its user-friendly interface, the availability of Fourier coefficient calculations, the ability to view the function graph and the Fourier series graph in one coordinate system. Due to the Internet resource [12] we get the display of power spectrum graph, which point references are calculated by FFT algorithm. The main feature among all of the above mentioned is the ability to select the input signal (file, record, function, etc.). However, despite the user-friendly and easy-to-understand interface, sufficient functional availability, given mathematical packets and online-calculators the processing of functions with variable period is not carried out.

The history of writing and developing software tools of computer mathematics shows that they are based on data models of to be processed, and the corresponding algorithms developed on the basis of models. Software is essentially the translation of the above mentioned algorithms into the language of computing means. As for the Fourier analysis of periodic functions with constant period, the algorithms for calculating Fourier coefficients are fundamental for the software. Taking this into account it is obvious that for the software tools of the Fourier analysis of the PFVP the theory and algorithms for calculating Fourier coefficients are required. For this purpose, let us consider some questions of the PFVP theory such as the methods of analytic assignment of such functions

3. Analytical assignment of functions with variable period. Let us give some examples of analytical assignment of functions with variable period.

3.1 Trigonometric functions $\sin x^{\alpha}$, $\cos x^{\alpha}$, $tg\, x^{\alpha}$, $ctg\, x^{\alpha}$, $\alpha > 0, \alpha \neq 1$. The first two of these functions – $\sin x^{\alpha}$, $\cos x^{\alpha}$ or their generalization $A\sin\left(kx^{\alpha} + \phi\right)$, $A\cos\left(kx^{\alpha} + \phi\right)$ are widely used. For example, the function with variable period is $f_1(x) = \sin x^{3/4}$ (Fig. 2), its variable period $T(x) = -x + \left(x^{3/4} + 2\pi\right)^{4/5}$.

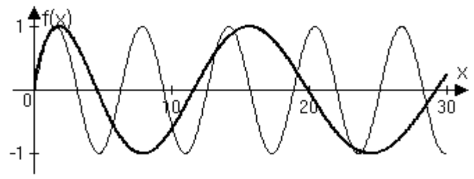

Fig. 2. Function $f_1(x) = \sin x^{3/4}$ **(thick line),** $f_2(x) = \sin x$ **(thin line)**

3.2 The exponential function: $f(x) = a^{g(x)}$, where the number $a > 0$, $g(x)$ – is the trigonometric function with variable period, in most cases $g(x) = \sin x^{\alpha}, \alpha > 0, \alpha \neq 1$, $g(x) = \cos x^{\alpha}$.

3.3 The power function: $f(x) = (g(x))^{a}$. For example, the power function is $f(x) = \left(\sin x^{4/3} + 1/2\right)^2$ (Fig. 3), its variable period is $T(x) = -x + \left(x^{4/3} + 2\pi\right)^{3/4}$.

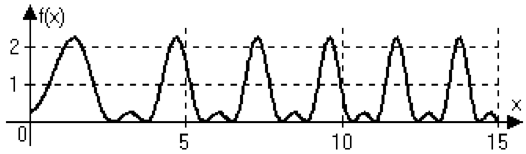

Fig. 3. The graph of function $f(x) = \left(\sin x^{4/3} + 1/2\right)^2$

Another example is the superposition of the power function and the “Fractional portion” function: $f(x) = \left\{x^{3/5}\right\}^2$. The graph of this function is shown in Fig. 4 with variable period $T(x) = -x + \left(x^{3/5} + 1\right)^{5/3}$.

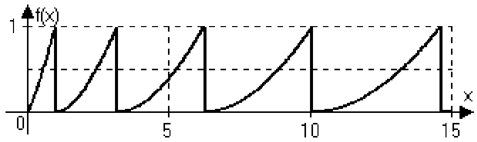

Fig. 4. The graph of function $f(x) = \left\{x^{3/5}\right\}^2$

3.4 The signum of the function: $f(x) = sign\, g(x)$, $g(x)$ – is trigonometric function with variable period. For example, the function $f(x) = sign\left(\sin x^{5/7}\right)$. Its graph – is the periodical fluctuations of the rectangular shape

(Fig. 5), but already with variable period $T(x) = -x + \left(x^{5/7} + 2\pi\right)^{7/5}$.

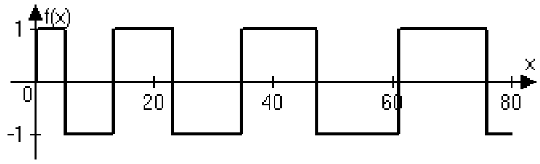

Fig. 5. The graph of function $f(x) = sign\left(\sin x^{5/7}\right)$

3.5 Fractional portion: $f(x) = \{g(x)\}^{\alpha}$, where $a > 0$, $\{\bullet\}$ – fractional portion, $g(x)$ – some nonlinear continuous increasing (decreasing) function.

Example. Superposition of the logarithmic function and the "fractional portion" function: $f(x) = \{\log_c x\},\ x \ge 1, c > 1$. For this function its variable period is $T(x) = x(c-1)$. At $c = 3$ the graph of function $f(x) = \{\log_3 x\}$ is shown in Fig. 6, and its period $T(x) = 2x$, i.e. is linear function kx with coefficient $k = 2$.

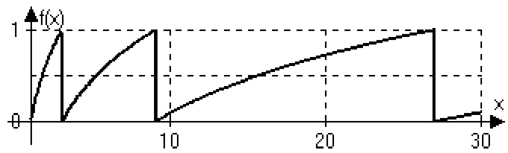

Fig. 6. The graph of function $f(x) = \{\log_3 x\}$

4. Fourier series of PFVP and its coefficients. The expansion of functions in the Fourier series is rather powerful tool for investigating the functions, primarily periodic ones. The formulas for determining the series coefficients are the basis of Fourier series development. Let us remind that if $f(x)$ is the periodic function with period T, then for its Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\frac{2\pi}{T}x + b_n \sin n\frac{2\pi}{T}x\right)$ the coefficients are determined by the formulas $a_0 = \frac{2}{T}\int_0^T f(x)dx$, $a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{2\pi}{T}xdx$, $b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{2\pi}{T}xdx$.

In the case of PFVP, the task of constructing Fourier series and determining their coefficients is much more complicated. Let us consider only one isolated case of this problem. We choose the system of trigonometric functions

$$\sin nx^{\alpha},\ \cos nx^{\alpha},\ \alpha > 0,\ \alpha \ne 1,\ n = 1,2,..., \tag{1}$$

Note that when $\alpha = 1$ we get the well-known system of trigonometric functions $\sin nx, \cos nx, n = 1,2,...$, orthogonal at the arbitrary interval $[x, x + 2\pi]$ with the length 2π. As far as system (1) is concerned, its variable period is determined by the formula

$$T(x) = -x + \left(x^{\alpha} + 2\pi\right)^{1/\alpha}. \tag{2}$$

According to [4] system (1) is orthogonal with the weight function and its orthogonality interval is the arbitrary segment $[x, x + T(x)], x \ge 0$. Taking into account (2), the orthogonality interval, is $[x, x + T(x)] = \left[x, \left(x^{\alpha} + 2\pi\right)^{1/\alpha}\right]$, i.e. its length is variable and equals $\left(x^{\alpha} + 2\pi\right)^{1/\alpha} - x$. It is easy to notice, that at $\alpha = 1$ the length of the orthogonality interval is equal 2π.

It is shown in [4] that the norm of each of the system functions (1) is equal to $\sqrt{\pi}$: $\left\|\sin nx^{\alpha}\right\| = \left\|cosnx^{\alpha}\right\| = \sqrt{\pi}$. This results from $\left\|\sin nx^{\alpha}\right\|^2 = \alpha\int_x^{x+T(x)} x^{\alpha-1}\sin^2 nx^{\alpha}dx = \pi$, $\left\|\cos nx^{\alpha}\right\|^2 = \alpha\int_x^{x+T(x)} x^{\alpha-1}\cos nx^{\alpha}dx = \pi, n = 1,2,\cdots$

Let us proceed to the problem of Fourier series of PFVP. Let us assume $f(x)$ – is the periodic function with variable period, whose variable period coincides with the variable period of the trigonometric system (1), i.e. is determined by formula (2). Fourier series $f(x)$ in this case is expressed as the sum

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx^{\alpha}+b_k\sin kx^{\alpha}\right). \tag{3}$$

It is shown in [13] that under the above mentioned conditions, Fourier coefficients for this series are determined by formulas

$$a_0=\frac{\alpha}{\pi}\int_{\tau}^{\tau+T(\tau)}x^{\alpha-1}f(x)dx,\ a_k=\frac{\alpha}{\pi}\int_{\tau}^{\tau+T(\tau)}x^{\alpha-1}f(x)\cos kx^{\alpha}dx,\ b_k=\frac{\alpha}{\pi}\int_{\tau}^{\tau+T(\tau)}x^{\alpha-1}f(x)\sin kx^{\alpha}dx \tag{4}$$

For Fourier series of PFVP there is Bessel inequality $\frac{a_0^2}{2}+\sum_{k=1}^{n}a_k^2+b_k^2\le\frac{1}{\pi}\|f(x)\|^2$, where norm is $\|f(x)\|^2=(f(x),f(x))=\alpha\int_{x}^{x+T(x)}x^{\alpha-1}f^2(x)dx$, i.e. determined by the scalar product taking into account the weight function $\rho(x)=\alpha x^{\alpha-1}$. For example, for $f(x)=sign\left(\sin x^{5/7}\right)$ (Fig. 8) its variable period is $T(x)=-x+\left(x^{5/7}+2\pi\right)^{7/5}$. Let assume that the left point of the integration interval is $\tau=20$. At this point the period is $T(20)=-20+\left(20^{5/7}+2\pi\right)^{7/5}\approx 23.4095$. Then, in order to find Fourier coefficients, integration should be carried out on the interval $[20,\ 20+T(20)]\approx[20,\ 43.4095]$. The coefficient calculation results are shown in Table 1.

Table 1

The coefficient calculation results

Interval $[20, 43.4095]$	Function with variable period $f(x)=sign\left(\sin x^{5/7}\right)$									
$a_k,\ k=\overline{0,9}$	0.0003	-0.0082	-0.0003	-0.0082	-0.0003	-0.0082	-0.00001	-0.0082	-0.0002	-0.0082
$b_k,\ k=\overline{1,10}$	-	1.2732	0.0000	0.4244	0.0000	0.2546	0.0000	0.1818	0.0000	0.1414
	$\frac{a_0^2}{2}+\sum_{k=1}^{19}\left(a_k^2+b_k^2\right)=1.9588$					$1/\pi\,\|f(x)\|^2=1.9999$				

Fig. 7 represents the graph of Fourier finite series function $f(x)=sign\left(\sin x^{5/7}\right)$ and the function graph. Comparing these graphs, it can be stated that Fourier finite series $\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx^{5/7}+b_k\sin kx^{5/7}\right)$ at $n=9$ displays the form of function.

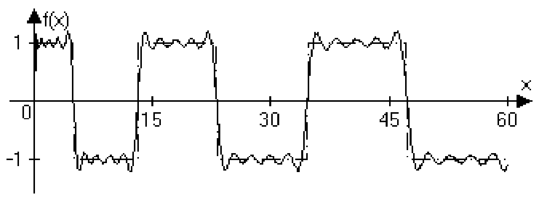

Fig. 7. Function $f(x)=sign\left(\sin x^{5/7}\right)$, $x\in[0,60]$, **(dotted line) and its Fourier finite series (full line)**

5. Software "Fourier-analysis of functions with variable period''. The availability of the analytical assignment of functions with variable period and algorithm for obtaining of Fourier series coefficients of the PFVP provide the opportunity to develop software performing the Fourier analysis of the PFDP.

The interface of the program involves introduction by user the input function for processing, as well as parameters for display and calculations. For our purpose the dialog window completely provides Fourier analysis performance, since it displays the input data for analysis, the graph of the investigated function and its approximation function as the result of the analysis, as well as the calculated Fourier series coefficients and the amplitude spectrum graph.

Due to the developed software the user can enter the function with variable period, but with certain features of the input syntax. All variables and operations should be capitalized. For example, in basic mathematical operations such as: Cos(), Sin(); Pow("value", "power") exponentiation, we record X in braces "[X]". That is, cosine X raised to the third power should be recorded in the following way Cos(Pow([X], 3)). If the function input syntax is broken in the text field, then its plotting does not take place. After clicking the "Build a graph" button, the graphing of the investigated FVP is visualized. There is also an option "Smoothing the function", which smoothest the graph of the investigated FVP.

In order to make calculations and analysis, the user inputs the number of coefficients of Fourier series and the left point of the integration onset. The "Calculate" button provides calculation of Fourier series coefficients, their output in the form of the table, as well as the visualization of the amplitude spectrum graph (Fig. 8).

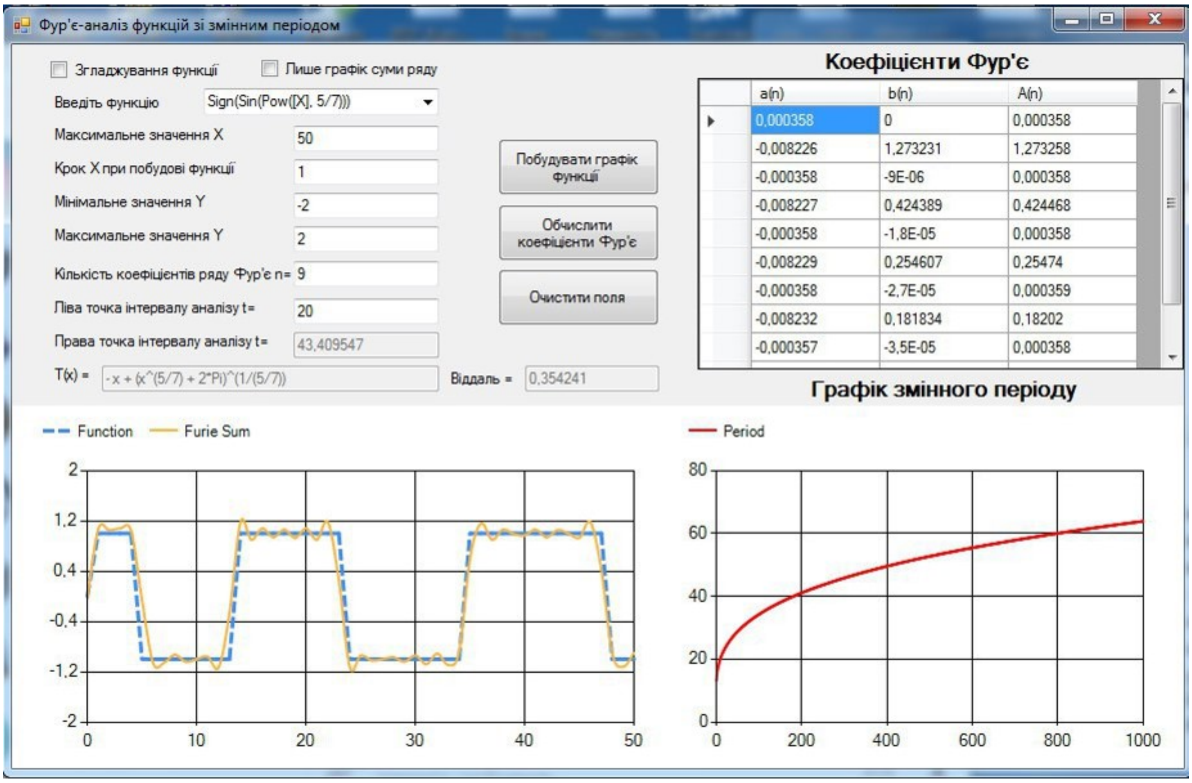

Fig. 8. Fourier-analysis of function $f(x) = sign\left(\sin x^{5/7}\right)$

Fourier coefficients of the investigated function are found and the Fourier series graph is constructed according to these coefficients sufficiently reflecting the behaviour of the function.

Conclusions. The carried out analysis of the modern mathematical software environments proved the existence of versatile possibilities for the investigation of periodic functions and the absence of such possibilities for periodic functions with variable period. The software program of Fourier analysis of functions with a variable period is developed. The obtained theoretical results are checked by means of computer experiment for one analytically assigned function with variable period. The obtained theoretical results are checked with the help of a computer experiment for one analytically given function with a variable period. For this purpose, Fourier coefficients of the investigated function are found, and their calculation was carried out by appropriate integration on one of the intervals which length coincided with the alternating period magnitude at the left point of the trigonometric system orthogonality interval. The graph of finite Fourier series constructed on the basis of these coefficients reflects the behaviour of the function perfectly well. The developed software product automatically calculates Fourier coefficients for functions with variable period and constructs the graphs of such functions and their finite Fourier series, which in turn provides the opportunity for further comprehensive investigations of functions with variable period.

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